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Multiple Self-Dual Strings on M5-Branes

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Abstract: We show how to define Chern-Simons matter theories with boundary. Rather than imposing boundary conditions, we introduce new boundary degrees of freedom from the beginning and show how they can be used to cancel the gauge non-invariance of the Chern-Simons action. We apply this method to the ABJM theory with boundary. By imposing also boundary conformal invariance, we determine the required boundary action. This result allows us to derive the action for the multiple self-dual strings living on M5-branes.

Keywords: Chern-Simons Theories, M-Theory, Anomalies in Field and String Theories, M(atrix) Theories.

Contents

1. Introduction

M-theory contains M2-branes and M5-branes. The understanding of the dynamics and interactions of these branes are some of the most important and mysterious aspects of M-theory. Recent progress has been made in the description of multiple M2-branes through the work of Bagger and Lambert [1–3] and Gustavsson [4]. The Bagger-Lambert (BL) theory [1–3] was originally motivated by trying to construct an action with manifest $\mathcal{N} = 8$ superconformal symmetry, based on a BPS equation postulated by Basu and Harvey [5]. This naturally led to an action with a non-abelian symmetry based on a Lie 3-algebra. However there is only one example of such a 3-algebra and it has been difficult to increase the rank of the gauge group.

Aharony, Bergman, Jafferis and Maldacena (ABJM) [6] proposed a $U(N)\times U(N)$ Chern-Simons gauge theory with levels k and $-k$ arises on the worldvolume of N M2-branes placed at the orbifold singularity $\mathbb{R}^8/\mathbb{Z}_k$. The theory allows arbitrary rank, but has only $\mathcal{N} = 6$ supersymmetry, although it is conjectured that supersymmetry is enhanced to $\mathcal{N} = 8$ for $k = 1, 2$. See for example [7] for some recent discussions on this issue.

The worldvolume theory on a flat M5-brane is given by a six dimensional (2,0) superconformal field theory. The massless excitations are given by the tensor multiplet which consists of a self-dual 2-form potential, five scalar fields and 8 fermions.

These fields are related to the breaking of the symmetries of the 11 dimensional supergravity, namely the gauge symmetry of the three-form potential, the 11 dimensional translational invariance, and the supersymmetries. For $(r+1)$ M5-branes parallel to each other, one obtains the A_r series of (2,0) theory [8,9]. The existence of these theories were first argued for [10] by considering type IIB string theory compactified on a K_3 with a 2-cycle shrinking to zero size and developing an A_r type singularity. The resulting IIB string theory is tensionless and is intrinsically non-perturbative, making it hard to study. These strings can also be viewed as the boundaries of open M2-branes which end on M5-branes. As tensionless self-dual strings are difficult to understand, one approach to the problem is to introduce a vev for one of the scalars which results in self-dual strings with finite tension. In the M-theory picture, turning on the vev corresponds to separating the M5-branes. A comprehensive review of this approach can be found in the thesis [11].

Just as strings can end on D-branes, M2-branes can end on M5-branes. It is natural to try to use the open M2-branes system to learn about the physics of the M5-brane or about the intersection itself, i.e. the self-dual string theory. Recently, by considering a system of M2-branes ending on an M5-brane with a constant C-field turned on, the quantum geometry on the M5-brane worldvolume has been derived [12]. In this paper, we will consider a system of open M2-branes and use it to learn about the physics of multiple self-dual strings.

For a single self-dual string, the action is given by a free part, plus a coupling to the self-dual tensor potential field, all constructed in a superconformal fashion [11]. In this paper, we are interested in constructing the bosonic part of the theory for multiple self-dual strings using the ABJM theory. In analogy to the usual matrix string theory [13], one expects to promote the string coordinates to matrices and introduce a self-interaction among them. What is new in the ABJM theory is the presence of a $U(N) \times U(N)$ twisted Chern-Simons action. Twisted here means the levels of the two gauge groups of the Chen-Simons action are opposite in sign. The action is not gauge invariant when there is a boundary. This leads to interesting new consequences for the dynamics of the self-dual strings theory.

The canonical way to deal with the gauge non-invariance for a standard Chern-Simons action is by imposing a boundary condition [14,15]. The boundary condition breaks gauge invariance at the boundary, and hence extra degrees of freedom reside there. Since there are two factors of $U(N)$ gauge group, the new extra degrees of freedom take values in each factor of the $U(N)$. This is well understood for ordinary Chern-Simons theory. For the ABJM theory, there are three points one has to be careful about. First, the Chern-Simons terms come with opposite levels. In [16,17], a straightforward application of the "imposing boundary condition" procedure was adopted. The authors found opposite kinetic terms for the new degrees of freedom

and so the boundary theory considered there is non-unitary. Secondly, the theory is no longer topological due to the presence of matter. Therefore even though one can impose boundary conditions on the gauge fields to make the gauge non-invariance of the action vanish, it is not clear in this approach how the expected degrees of freedom would arise (see the discussions in the last paragraph of sec 2.3). Finally, if we consider a boundary ABJM theory which preserves part of the bulk supersymmetry, the boundary degrees of freedom will have to form a supermultiplet of the boundary supersymmetry. This will provide strong constraints on the boundary degrees of freedom.

In this paper, we demonstrate a new way to treat the gauge non-invariance created by the boundary Chern-Simons action. For pure untwisted Chern-Simon theory, our methods gives identical results to the original treatments of [14, 15]. However our method is applicable also for twisted Chern-Simons theory with matter, e.g. the ABJM theory. We will see that this gauge non-invariant term is analogous to the gauge anomaly in many ways, so we will call this an anomaly. We find that apart from imposing a boundary condition, one can do away with the anomaly by introducing a set of extra boundary degrees of freedom whose action has a variation which cancels the anomaly. Not surprisingly, the required action is the Wess-Zumino term [18]. Our construction is modeled after the original construction of Wess and Zumino [18]. The extra degrees of freedom play the role of the Nambu-Goldstone bosons. In addition to gauge invariance, the boundary ABJM theory is also expected to be conformally invariant. This can be achieved by adding additional kinetic terms to the theory. This results in a Wess-Zumino-Witten (WZW) theory.

The plan of the paper is as follows. In section 2, we first review the original treatment of [14, 15] for a pure untwisted boundary Chern-Simons action. Then we explain our new method in section 2.2. A comparison of the two methods is given in section 2.3, where we demonstrate that the two are equivalent for the pure Chern-Simon case. In section 3.1, we apply our construction to the bosonic ABJM theory with a boundary. We argue for and identify the extra degrees of freedom that must be present on the boundary of the open M2-branes system. The gauge invariant and conformal invariant bosonic action is determined. In section 3.2 we take into account the boundary supersymmetry and describe how the bosonic field content has to be enlarged. The resulting supersymmetric WZW model for these degrees of freedom is presented. Then, in section 3.3, by considering a specific configuration of open membranes suspended between two M5-branes, we determine the action for a system of multiple self-dual strings. The paper concludes with discussions in section 4.

2. Chern-Simons Theory and Boundary Action

Consider a Chern-Simons theory with gauge group G on a 3-dimensional manifold M

$$S_{CS} = \frac{k}{4\pi}\int_M \omega_3(A) = \frac{k}{4\pi}\int_M \mathrm{Tr}(AdA + \frac{2}{3}A^3), \tag{2.1}$$

where A^3 denotes $A \wedge A \wedge A$ etc and $A = dx^\mu A_\mu$ is a Lie algebra valued one-form. When M is closed, the theory is gauge invariant and topological. In general the Chern-Simons form $\omega_{2n+1}(A)$ satisfies

$$\mathrm{Tr}F^{n+1} = d\omega_{2n+1}(A), \quad \text{where } F = dA + A^2. \tag{2.2}$$

Under an infinitesimal gauge transformation

$$\delta_\alpha A = d\alpha + [A, \alpha], \quad \delta_\alpha F = [F, \alpha], \tag{2.3}$$

the Chern-Simons form transforms as

$$\delta_\alpha \omega_{2n+1}(A) = d\omega^1_{2n}(A; \alpha). \tag{2.4}$$

Explicit expressions for $\omega_{2n+1}(A)$ and $\omega^1_{2n}(A; \alpha)$ can be computed using the Cartan homotopy operator, see [19] for details. For example,

$$\omega^1_2(A; \alpha) = \mathrm{Tr}(\alpha dA). \tag{2.5}$$

To study the theory in the presence of a boundary ∂M, we note that due to (2.4) the action S_{CS} is not gauge invariant, but its variation is a boundary term. Hence the gauge invariance of the action S_{CS} is broken at the boundary. These boundary terms will vanish with appropriate boundary conditions [14, 15]. We will review this approach in the next subsection. Alternatively they can be cancelled by the variations of additional degrees of freedom. We will explain this new approach in subsection 2.2.

2.1 Imposing boundary conditions

Consider an arbitrary infinitesimal variation of A, the variation of the Chern-Simons action gives

$$\delta S_{CS} = \frac{k}{2\pi}\int_M \mathrm{Tr}(\delta A F) + \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\left(\delta A A\right). \tag{2.6}$$

The bulk term gives the equation of motion $F = 0$. For the boundary term to vanish, one can impose a boundary condition on A and only allow variations which preserve the boundary condition. The most general condition is a linear relation between the two boundary components of A. Each choice of the boundary condition

corresponds to a definition of the boundary theory. In general, one can divide the possible boundary conditions into different inequivalent classes, each corresponding to a possible definition of the boundary theory.

For physical applications, we consider manifolds of the form $M = \mathbb{R} \times \Sigma$, where the noncompact direction $\mathbb{R}$ is interpreted as time and $\partial\Sigma \neq 0$ [15]. Let's consider explicitly a boundary at $x^2 = 0$ and choose the boundary condition $A_0 = 0$. With this boundary condition, one can write

$$S_{CS} = \frac{k}{2\pi} \int_M \mathrm{Tr}(\epsilon^{ij} F_{ij} A_0 - \frac{1}{2}\epsilon^{ij} A_i \dot{A}_j), \tag{2.7}$$

where we have integrated by parts the terms involving a derivative of A_0, whose resulting boundary terms vanishing either due to the usual asymptotic boundary conditions, or because A_0 vanishes on ∂M. The result is that A_0 only appears linearly in (2.7), and is therefore a Lagrange multiplier, imposing the constraint $F_{12} = 0$. This means that we can write

$$A_i = U^{-1}\partial_i U \quad \text{for} \quad i = 1, 2 \tag{2.8}$$

for $U \in G$. If we then substitute this back into the action (2.7) we find

$$S = -\frac{k}{8\pi} \int_{\partial M} \mathrm{Tr}\left(U^{-1}\partial_0 U U^{-1}\partial_1 U\right) - \frac{k}{12\pi} \int_M \mathrm{Tr}\left(U^{-1} dU\right)^3. \tag{2.9}$$

Note that a "chiral kinetic term" is obtained for U. A few remarks follow.

1. Note that this result is essentially the same for any allowed boundary conditions for A. However, the form of the boundary (kinetic) term does depend on whether we choose a timelike, spacelike or lightlike combination of components of A to vanish on the boundary. For example, if we had taken the boundary condition $A_1 = 0$ instead of $A_0 = 0$, then the sign of the kinetic term would be reversed. Furthermore, if we instead chose a light-like combination $A_0 \pm A_1 = 0$ on the boundary, we would get a "conventional kinetic term" $(U^{-1}\partial_\mu U)^2$ on the boundary, with the overall sign depending on which light-like direction we chose.

   So, choosing appropriate boundary conditions for A, we arrive at the well-known WZW action

   $$S_{WZW}[U] = -\frac{k}{8\pi} \int_{\partial M} \mathrm{Tr}(U^{-1}\partial_\mu U)^2 - \frac{k}{12\pi} \int_M \mathrm{Tr}\left(U^{-1} dU\right)^3, \tag{2.10}$$

   where the metric is $\eta_{00} = -1 = -\eta_{11}$. This action describes the dynamics for the field U living on the boundary ∂M.

2. We remark that while the above derivation is classical, it was argued in [15] that there is no non-trivial Jacobian introduced in the path integral by the change of variables (2.8). Therefore the action in terms of U is equivalent to the original Chern-Simons action with boundary conditions on A.

3. We note that although the boundary conditions break the original gauge symmetry, the resulting WZW action (2.10) does have a symmetry [20]. In fact the action is invariant under chiral transformations of the form

$$U \to \Omega(z) U \tilde{\Omega}(\bar{z}) \tag{2.11}$$

where $z = x^0 + ix^1$ and $\Omega, \tilde{\Omega} \in G$. This chiral $G \times G$ symmetry is generated by the currents ($\partial = \partial_z$, $\bar{\partial} = \partial_{\bar{z}}$)

$$J = \frac{k}{\pi} U^{-1} \partial U, \quad \bar{J} = \frac{k}{\pi} \bar{\partial} U U^{-1}, \tag{2.12}$$

which are chiral, $\partial \bar{J} = \bar{\partial} J = 0$ and satisfy a Kac-Moody algebra

$$\begin{aligned} [J^a(z_1), J^b(z_2)] &= i f^{ab}_c J^c(z_1) \delta(z_1 - z_2) - i\frac{k}{\pi} \delta'(z_1 - z_2) \delta^{ab}, \\ [\bar{J}^a(\bar{z}_1), \bar{J}^b(\bar{z}_2)] &= i f^{ab}_c \bar{J}^c(\bar{z}_1) \delta(\bar{z}_1 - \bar{z}_2) + i\frac{k}{\pi} \delta'(\bar{z}_1 - \bar{z}_2) \delta^{ab}. \end{aligned} \tag{2.13}$$

We remark that the Kac-Moody symmetry is not part of the original gauge symmetry of the Chern-Simons action.

4. We also note that the action (2.10) is conformal – for this to hold at the quantum level the relative coefficient between the bulk and boundary terms is fixed, as above, up to a sign [21]. The fact that a conformal field theory arises on the boundary can be understood as follows: we start out with the bulk Chern-Simons theory which is topological, i.e. invariant under arbitrary variations of the metric. By imposing a boundary condition, a conformal structure is fixed on the boundary manifold. Therefore the boundary theory can only be invariant under arbitrary variations of the metric which preserve this conformal structure. Hence the boundary theory is conformally invariant.

2.2 Boundary degrees of freedom

Another point of view is that to render the theory gauge invariant, there should arise additional physical degrees of freedom at the boundary. The total action should be chosen so that the gauge variation of the additional terms cancels the gauge variation of the original action S_{CS}.

To do this, let us note it is well known that the object $\omega^1_{2n}(A;\alpha)$ is related to the chiral anomaly in a $2n$-dimensional gauge theory. One of the consequences of

the anomaly is that the low energy effective theory of the Nambu-Goldstone bosons admits a Wess-Zumino term [18]. For the simplest case where the whole gauge group G is broken by an anomaly, the Nambu-Goldstone bosons $g = e^{-\xi}$ are G-valued and the Wess-Zumino effective action is given by [18],

$$W[\xi, A] := \int_0^1 dt\, G_\xi[A_t], \tag{2.14}$$

where A_t is the gauge field $A_t := e^{t\xi} A e^{-t\xi} + e^{t\xi} d e^{-t\xi}$ and $G_\alpha[A] = \int \omega^1_{2n}(A;\alpha)$ is the anomaly. The effective action was constructed to reproduce the anomaly: $\delta_\alpha W[\xi, A] = G_\alpha[A]$. It is not difficult to show that [22] [1]

$$W[\xi, A] = \int_M (-\omega_{2n+1}(A^g) + \omega_{2n+1}(A)), \tag{2.15}$$

where M is a $(2n+1)$-dimensional manifold whose boundary ∂M is equal to the $2n$-dimensional spacetime and A^g is the transform of A under a finite gauge transformation

$$A^g := g^{-1} A g + g^{-1} dg. \tag{2.16}$$

Note that since $d\omega_{2n+1}(A) = \mathrm{tr} F^{2n+2}(A)$, the integrand is closed and so (2.15) actually defines a $2n$-dimensional action on ∂M. If one expands (2.15) around $A = 0$, then one finds the WZW term [21]

$$\int_M \mathrm{Tr}\, (g^{-1} dg)^{2n+1}. \tag{2.17}$$

Back to our case of a 3-dimensional Chern-Simons theory with a boundary. The gauge non-invariance of the Chern-Simons action is given by $\frac{k}{4\pi} \int_{\partial M} \omega^1_2(A;\alpha)$ and is precisely of the same form as the chiral anomaly reviewed above. Therefore in order to restore gauge invariance, one can introduce additional degrees of freedom g that live on the boundary (this plays the role of the Nambu-Goldstone bosons) with the following action

$$S_{Bdry} := \frac{k}{4\pi} \int_M \left[\omega_3(A^g) - \omega_3(A)\right], \tag{2.18}$$

so as to cancel against the gauge noninvariant terms resulting from the Chern-Simons action. Since S_{Bdry} defines an action on the boundary manifold ∂M, one can add it to the Chen-Simons term. Under a gauge transformation with parameter h,

$$A^g \to (A^g)^h = A^{hg}. \tag{2.19}$$

[1] The paper [22] deals with the general situation of a gauge theory with gauge group G and with a subgroup H such that the associated currents are anomaly free. In this case the Wess-Zumino term can be constructed from the Nambu-Goldstone bosons which are valued in the coset space G/H. To apply to the present case with trivial H, we just need to set $A_h = 0$ in the formula in eqn (115) of appendix 2 there.

Therefore A^g and hence $\omega_3(A^g)$ will be gauge invariant under the combined transformation

$$A \to A^h \quad , \quad g \to h^{-1} g. \tag{2.20}$$

The term S_{Bdry} thus has a gauge variation which cancels exactly that of S_{CS} and so the total action

$$S_T := S_{CS} + S_{Bdry} \tag{2.21}$$

is gauge invariant.

Note that the resulting action is not unique. We are free to add any further gauge-invariant boundary terms. A natural choice is to try to preserve as much as possible of the original symmetries of the bulk action. Since the original action was topological, we can at least try to preserve conformal invariance on the boundary. Now, if we set the gauge field $A = 0$ we are left with

$$S_{Bdry}[A=0] = -\frac{k}{12\pi} \int_M \mathrm{Tr}\left(g^{-1} dg\right)^3 . \tag{2.22}$$

We can introduce a boundary term giving a kinetic term for the boundary field g, resulting in the well-known WZW conformal field theory

$$S_{WZW}[g] = -\frac{k}{8\pi} \int_{\partial M} \mathrm{Tr}(g^{-1}\partial_\mu g)^2 - \frac{k}{12\pi} \int_M \mathrm{Tr}\left(g^{-1} dg\right)^3 . \tag{2.23}$$

Restoring the gauge field A we can maintain both gauge and conformal invariance by adding boundary terms as above, provided we replace the partial derivative ∂_μ with a covariant derivative $D_\mu = \partial_\mu + A_\mu$. We therefore have the total boundary action

$$\begin{aligned} S_{Bdry} &= -\frac{k}{8\pi} \int_{\partial M} \mathrm{Tr}(g^{-1} D_\mu g)^2 + \frac{k}{4\pi} \int_M [\omega_3(A^g) - \omega_3(A)] \\ &= S_{WZW}[g] + \frac{k}{4\pi} \int_{\partial M} \partial_+ g g^{-1} A_- - \frac{k}{8\pi} \int_{\partial M} A_\mu^2, \end{aligned} \tag{2.24}$$

where $\partial_\pm := \partial_0 \pm \partial_1$ and the boundary metric is $\eta_{00} = -1 = -\eta_{11}$. The 2-dimensional action (2.24) describes the dynamics of the group-valued degrees of freedom g in interaction with an external gauge field. We remark that it is not the same as the standard gauged WZW action

$$S_{\mathrm{gauged}\,WZW} := S_{WZW} + \frac{k}{4\pi} \int_{\partial M} (A_+ \partial_- g g^{-1} - A_- g^{-1} \partial_+ g + A_+ g A_- g^{-1} - A_- A_+). \tag{2.25}$$

In particular, our S_{Bdry} is invariant under the gauge transformation $A_\mu \to h^{-1} A_\mu h + h^{-1} \partial_\mu h$, $g \to h^{-1} g$; while $S_{\mathrm{gauged}\,WZW}$ is invariant under a different gauge transformation for g: $g \to h^{-1} g h$.

Adding the boundary action S_{Bdry} to S_{CS}, the total action is

$$S_T = -\frac{k}{8\pi}\int_{\partial M}\mathrm{Tr}(g^{-1}D_\mu g)^2 + \frac{k}{4\pi}\int_M \omega_3(A^g). \tag{2.26}$$

Note that above we introduced the kinetic term for g with a specific coefficient so that the action in conformal at the quantum level, not just classically. However, this requirement does not fix the sign of the kinetic term and so for any given sign of k, we are free to choose the kinetic term to have the correct sign. This observation will be important when describing the boundary extension of the ABJM action where there are two Chern-Simons terms with opposite levels.

Let us now derive the boundary conditions of this action, which are imposed on-shell as boundary equations of motion. This can be derived by concentrating on the boundary contributions to δS. We note that some care must be taken. Since A^g contains derivatives of g, the variation δA^g in the bulk, when expressed in terms of δA and δg will also give a boundary contribution involving δg.

Consider first varying A. Since $g^{-1}D_\mu g = A^g_\mu$, the variation of the boundary term in (2.26) is

$$-\frac{k}{4\pi}\int_{\partial M}\mathrm{Tr}\left(A^{g\,\mu}\delta A^g_\mu\right); \tag{2.27}$$

while, noting that $\delta A^g = g\delta A g^{-1}$ does not contain derivatives of δA, the boundary contribution in the variation of the bulk term in (2.26) is contained in

$$\frac{k}{4\pi}\int_M \mathrm{Tr}\left[A^g d\left(\delta A^g\right)\right] \sim -\frac{k}{4\pi}\int_{\partial M}\mathrm{Tr}\left(A^g\delta A^g\right). \tag{2.28}$$

Combining these two contributions we find the boundary term

$$-\frac{k}{4\pi}\int_{\partial M}\mathrm{Tr}\left(A^g_+\delta A^g_-\right). \tag{2.29}$$

Therefore we obtain from δA the boundary condition

$$A^g_+ = 0 \tag{2.30}$$

which is equivalent to

$$A_+ = -(\partial_+ g)g^{-1}. \tag{2.31}$$

Next we consider the variation of g, noting that now

$$\delta A^g = -g^{-1}\delta g g^{-1}(Ag + dg) + g^{-1}A\delta g + g^{-1}d(\delta g) \tag{2.32}$$

contains derivatives of δg. The variation of the boundary term is

$$-\frac{k}{4\pi}\int_{\partial M}\mathrm{Tr}\left(A^{g\,\mu}\delta A^g_\mu\right), \tag{2.33}$$

while the variation of the bulk term gives the following contribution to the boundary variation

$$\begin{aligned}
&\frac{k}{4\pi}\int_M \mathrm{Tr}\left(\delta A^g dA^g + A^g d\left(\delta A^g\right) + 2A^g A^g \delta A^g\right) \\
&= -\frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\left(A^g \delta A^g\right) + \frac{k}{2\pi}\int_M \mathrm{Tr}\left[\left(dA^g + A^g A^g\right)\delta A^g\right].
\end{aligned} \tag{2.34}$$

The explicit boundary terms cancel that of (2.33) since $A_+^g = 0$; while the final bulk term gives a boundary contribution from $\delta A^g \sim g^{-1}d(\delta g)$,

$$-\frac{k}{2\pi}\int_{\partial M} \mathrm{Tr}\left(F^g g^{-1}\delta g\right) = -\frac{k}{2\pi}\int_{\partial M} \mathrm{Tr}\left(g^{-1}F\delta g\right). \tag{2.35}$$

Therefore we find from δg the boundary condition $F_{\mu\nu} = 0$, $\mu, \nu = 0, 1$. Together with (2.31), we obtain

$$A_\mu = -(\partial_\mu g)g^{-1}, \quad \mu = 0, 1. \tag{2.36}$$

Summarizing, the result is that in M we have equations of motion $F = 0$ so that A is pure gauge. The boundary equations of motion, give the boundary conditions $A = -dgg^{-1}$, and so we have $A = -dgg^{-1}$ in M where g is an arbitrary extension of the boundary field g into M.

2.3 Comparing the two approaches

To compare with the alternative method of imposing a boundary condition for A, we note that A_+ appears linearly in S_T and no derivatives of it is involved. This can be seen by rewriting the Chern-Simons action in the following form

$$S_{CS} = \frac{k}{4\pi}\int_M \mathrm{Tr}\ A_+F_{2-} + \frac{k}{8\pi}\int_M \mathrm{Tr}\ (A_2\partial_+A_- - A_-\partial_+A_2) - \frac{k}{8\pi}\int_{\partial M} \mathrm{Tr} A_-A_+, \tag{2.37}$$

where $A_\pm = A_0 \pm A_1$, $dx^\pm = (dx^0 \pm dx^1)/2$ etc. The last term in (2.37) cancels precisely the last term in (2.24) and so

$$\begin{aligned}
S_T = S_{WZW}[g] &+ \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\ \partial_+ g g^{-1} A_- + \frac{k}{4\pi}\int_M \mathrm{Tr}\ A_+F_{2-} \\
&+ \frac{k}{8\pi}\int_M \mathrm{Tr}\ (A_2\partial_+A_- - A_-\partial_+A_2).
\end{aligned} \tag{2.38}$$

As a result, A_+ is a (bulk) Lagrange multiplier and can be integrated out, imposing the constraint $F_{2-} = 0$ in M. Solving this constraint by writing $A_2 = \lambda^{-1}\partial_2\lambda$ and $A_- = \lambda^{-1}\partial_-\lambda$, we find

$$S_T = S_{WZW}[g] + S_{WZW}[\lambda] + \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\left(\partial_+ g g^{-1}\lambda^{-1}\partial_-\lambda\right) = S_{WZW}[\lambda g], \tag{2.39}$$

where we have used the Polyakov-Wiegmann identity in the last step.

Therefore with the identification $U = \lambda g$, we finally arrive at the same action as obtained by the boundary condition approach. The degree of freedom λ which arises from the gauge field A and the boundary degree of freedom g naturally combine into the variable U which appears in the boundary condition approach. Physically this is expected since the Chern-Simons theory without boundary is topological and describes a flat-connection over M. This corresponds to a single gauge function U when projected to the boundary.

Note that by imposing a boundary condition, we break the gauge symmetry, while by adding boundary degrees of freedom g we preserve the gauge symmetry. However as mentioned already, after integrating out the Lagrange multiplier, g and λ combine into a single variable $U = \lambda g$. This variable is a gauge singlet (since gauge transformations transform $g \to h^{-1}g$ together with $\lambda \to \lambda h$) and so the gauge symmetry is not apparent when we use the description in terms of U.

In the next section we will deal with the boundary ABJM theory with new features that one needs to be careful with. In addition to a twisted $U(N) \times U(N)$ Chern-Simons action, the presence of matter brings in a new complication. Due to the matter-gauge couplings, the action is quadratic in the gauge field components after imposing a boundary condition. This is in contrast with the pure Chern-Simons case where the action with the boundary condition imposed is linear in one of the gauge fields and so allows one to trade the remaining gauge fields in terms of the new degrees of freedom U as in (2.8). In the ABJM case, one cannot see the emergence of new degrees of freedom this way. If one does try to integrate out some of the gauge fields as in the original boundary condition approach for the pure Chern-Simons action, one finds that the remaining gauge field is dynamical and the action is nonlocal. The resulting theory is complicated and highly nontrivial. It is hard to see how to proceed with this approach. On the other hand, one can still follow the approach of adding degrees of freedom to derive the action for the boundary theory. In this case, one will still have the added boundary gauge degrees of freedom g and the degrees of freedom λ which arises from the gauge field A. However, they do not combine into a single gauge invariant variable anymore. The $U(N) \times U(N)$ gauge symmetry is therefore manifest and will play an important role. Another new feature is the presence of supersymmetry at the boundary. This provides additional constraints on the form of the boundary degrees of freedom and their action.

3. Action for Multiple Self-Dual Strings on M5-Branes

3.1 Bosonic ABJM open membrane theory with boundary

Let us apply our above construction to study the ABJM theory in the presence of a

boundary. The bosonic part of the action for the ABJM theory is given by

$$S_{ABJM} = S_{CS} + S_C \tag{3.1}$$

where

$$S_{CS} = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A^{(1)}dA^{(1)} + \frac{2}{3}{A^{(1)}}^3\right) - \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A^{(2)}dA^{(2)} + \frac{2}{3}{A^{(2)}}^3\right), \tag{3.2}$$

$$\begin{aligned} S_C = -\int_M \mathrm{Tr}\big(D_M C_I^\dagger D^M C^I\big) - \frac{4\pi^2}{3k^2}\int_M \mathrm{Tr}\Big(&C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K \\ &+ 4C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger - 6C^I C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger\Big) \end{aligned} \tag{3.3}$$

and the level k is a positive integer. Here $A^{(1)}$ and $A^{(2)}$ are the gauge potentials for the two $U(N)$ factors in the gauge group. The matter fields C^I $(I = 1, 2, 3, 4)$ are in the bifundamental representation $(N, \bar{N})$ and the covariant derivative acts as $D_M C^I = \partial_M C^I + A_M^{(1)} C^I - C^I A_M^{(2)}$, where $M = 0, 1, 2$.

In the absence of a boundary, the action is gauge invariant under:

$$A^{(i)} \rightarrow A^{(i)h^{(i)}} \quad , \quad C^I \rightarrow (h^{(1)})^{-1} C^I h^{(2)}. \tag{3.4}$$

But when there is a boundary, the Chern-Simons terms are not gauge invariant. Naively, one may try to fix this by imposing boundary conditions as in the previous section. However there is an important difference here since the gauge fields are coupled to the matter fields. As we discussed above, unlike the case for pure Chern-Simons action, the bulk action is not linear in any component of the gauge fields. Therefore, even after imposing boundary conditions, we will not be able to integrate out a Lagrange multiplier. If we did follow this method, we would end up with a more complicated action, and we would also not expect such an action to be equivalent at the quantum level. Therefore, we would be left with the original action subject to boundary conditions. The emergence of the needed degrees of freedom is obscure. Instead, the second approach of introducing new degrees of freedom allows us to explicitly preserve the desired symmetries of the bulk action, and interpret the final invariant action in terms of boundary degrees of freedom.

To maintain gauge invariance, we can add boundary terms of the form described in the previous section for each of the gauge fields $A^{(1)}$ and $A^{(2)}$. The combined action would then be gauge invariant. Since the ABJM theory also has conformal invariance, we should include the boundary kinetic terms also. As a result, the required boundary actions are

$$S_{Bdry,1} = -\frac{k}{8\pi}\int_{\partial M} \mathrm{Tr}(g^{-1} D_\mu^{(1)} g)^2 + \frac{k}{4\pi}\int_M (\omega_3(A^{(1)g}) - \omega_2(A^{(1)})) \tag{3.5}$$

and

$$S_{Bdry,2} = -\frac{k}{8\pi}\int_{\partial M}\mathrm{Tr}(\hat{g}^{-1}D_\mu^{(2)}\hat{g})^2 - \frac{k}{4\pi}\int_M(\omega_3(A^{(2)\hat{g}}) - \omega_2(A^{(2)})). \tag{3.6}$$

Note that although the bulk Chern-Simons terms for $A^{(1)}$ and $A^{(2)}$ differ by a relative sign, as previously mentioned the sign of the boundary kinetic terms in the actions (3.5) and (3.6) is not fixed by conformal invariance. We have therefore chosen the same sign for these terms so that we have conventional kinetic terms, and hence a well defined quantum field theory, for the boundary fields g and $\hat{g}$. More explicitly, we have

$$S_{Bdry,1}[g, A^{(1)}] = S^{(-)}_{WZW}[g] + \frac{k}{4\pi}\int_{\partial M}\partial_+ g g^{-1}A_-^{(1)} - \frac{k}{8\pi}\int_{\partial M}A_\mu^{(1)\,2}, \tag{3.7}$$

$$S_{Bdry,2}[\hat{g}, A^{(2)}] = S^{(+)}_{WZW}[\hat{g}] - \frac{k}{4\pi}\int_{\partial M}\partial_-\hat{g}\hat{g}^{-1}A_+^{(2)} - \frac{k}{8\pi}\int_{\partial M}A_\mu^{(2)\,2}, \tag{3.8}$$

where the WZW actions are defined by

$$S^{(\pm)}_{WZW}[h] := -\frac{k}{8\pi}\int_{\partial M}\mathrm{Tr}(h^{-1}\partial_\mu h)^2 \pm \frac{k}{12\pi}\int_M \mathrm{Tr}\left(h^{-1}dh\right)^3. \tag{3.9}$$

Adding the boundary actions (3.5) and (3.6) to the action for the boundary fields g and $\hat{g}$, the full gauge invariant and conformal invariant action describing a system of N ABJM open membranes is given by

$$\begin{aligned}
S_T = {} & \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A^{(1)}dA^{(1)} + \frac{2}{3}A^{(1)^3}\right) - \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A^{(2)}dA^{(2)} + \frac{2}{3}A^{(2)^3}\right) \\
& - \int_M \mathrm{Tr}\big(D_M C_I^\dagger D^M C^I\big) + \frac{4\pi^2}{3k^2}\int_M \mathrm{Tr}\Big(C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K \\
& \qquad + 4C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger - 6C^I C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger\Big) \\
& + S^{(-)}_{WZW}[g] + \frac{k}{4\pi}\int_{\partial M}\partial_+ g g^{-1}A_-^{(1)} - \frac{k}{8\pi}\int_{\partial M}A_\mu^{(1)\,2} \\
& + S^{(+)}_{WZW}[\hat{g}] - \frac{k}{4\pi}\int_{\partial M}\partial_-\hat{g}\hat{g}^{-1}A_+^{(2)} - \frac{k}{8\pi}\int_{\partial M}A_\mu^{(2)\,2} - \int_{\partial M}V_{\partial M}(C).
\end{aligned} \tag{3.10}$$

Here we have included a possible boundary interaction term $V_{\partial M}$ for the matter fields C^I. Due to conformal invariance, the boundary potential is quartic, $V_{\partial M} \sim \sum C^4$. Classically, it is

$$V_{\partial M} = \alpha^{IJKL}\mathrm{Tr}(C^I C_J^\dagger C^K C_L^\dagger). \tag{3.11}$$

The coefficients satisfy $\alpha^{IJKL\dagger} = \alpha^{JILK} = \alpha^{LKJI}$. Classically these coefficients are free. The requirement of quantum conformal invariance will provide further constraints on these coefficients. The form of $V_{\partial M}$ is further constrained if the open membranes theory has a boundary which preserves some amount of supersymmetry.

We can now see how this formulation encodes boundary conditions for the gauge fields, in the form of boundary equations of motion. To be more explicit we consider the boundary to be at $x^2 = 0$. It is important to note that variations of the gauge fields in S_C will contribute to the bulk equations of motion, but they will not give rise to any boundary terms. Therefore using the results from section 2.2, we obtain immediately the boundary condition from the variation $\delta A^{(1)}$:

$$A_+^{(1)g} = 0 \tag{3.12}$$

which is equivalent to

$$A_+^{(1)} = -(\partial_+ g)g^{-1}. \tag{3.13}$$

Clearly a similar analysis would follow for $A^{(2)}$, but the different relative sign between the bulk and boundary terms would give the result

$$A_-^{(2)\hat{g}} = 0, \tag{3.14}$$

or equivalently

$$A_-^{(2)} = -(\partial_- \hat{g})\hat{g}^{-1}. \tag{3.15}$$

Note that in the bulk action, compared to the Chern-Simons action there are variations of $A^{(i)}$, arising from the gauged kinetic terms for C. These do not modify the boundary terms, but change the bulk equations of motion, so that instead of $F^{(i)} = 0$ we have the equation of motion

$$\begin{aligned} \frac{k}{2\pi} F^{(1)} &= *C^I D C_I^\dagger - h.c. \\ \frac{k}{2\pi} F^{(2)} &= *D C_I^\dagger C^I - h.c. \end{aligned} \tag{3.16}$$

This is a direct consequence of the fact that, due to the presence of matter, the ABJM theory is not topological.

Next we consider the variations of g and $\hat{g}$. Since these boundary fields only appear in the modified Chern-Simons actions, their effect is described fully by the considerations of section 2.2. The boundary equations of motion therefore result in the boundary conditions

$$F_{\mu\nu}^{(i)} = 0, \qquad \mu, \nu = 0, 1. \tag{3.17}$$

Together with (3.13) and (3.15), this means

$$A_\mu^{(1)} = -(\partial_\mu g)g^{-1} \quad \text{and} \quad A_\mu^{(2)} = -(\partial_\mu \hat{g})\hat{g}^{-1}. \tag{3.18}$$

Finally, variations of C^I give the boundary condition

$$D_2 C^I = -U^I, \tag{3.19}$$

where U^I denotes the variation $\delta V_{\partial M}/\delta C_I^\dagger$. Using this and the consistency of (3.17) with (3.16) implies that

$$C^I U_I^{\ \dagger} - U^I C_I^{\ \dagger} = 0 \tag{3.20}$$

on ∂M. It is easy to check that (3.20) is satisfied for (3.11) and so the bulk equations of motion (3.16) are consistent with the boundary conditions (3.17).

3.2 $\mathcal{N} = (4,4)$ boundary ABJM theory

In the previous section we did not include supersymmetry in the discussion. If we consider a set of multiple M2-branes ending orthogonally on a M5-brane in flat space, the intersecting M2/M5 system will preserve a quarter of the supersymmetry, i.e. $\mathcal{N} = (4,4)$ in two dimensions. For $k > 2$ the C^4/Z_k orbifold will break the supersymmetry further, but we expect the boundary field content to be consistent with the $\mathcal{N} = (4,4)$ multiplet structure for any value of k, in the same way as the ABJM field content is consistent with $\mathcal{N} = 8$ supersymmetry in three dimensions. Since the bulk ABJM theory has only manifest $\mathcal{N} = 6$ supersymmetry, we expect the boundary action, upon imposing a suitable boundary condition on the C-fields which corresponds to the M5-brane, to have only manifest $\mathcal{N} = (3,3)$ supersymmetry.

First, let us look at the boundary condition for the C-fields. Following the $\mathcal{N} = (2,2)$ superspace construction of [24], it is convenient to introduce $SU(2)$ fields by writing $C^I = \{Z^I, W^{I^\dagger}\}$, $C_I^\dagger = \{Z_I^\dagger, W_I\}$, where the index I on the right hand side now runs from 1 to 2. Here Z's are in the representation $(N, \bar{N})$ and W's are in the representation $(\bar{N}, N)$. In this formulation, the ABJM theory posesses an $SU(4)_R$ global symmetry. A particular BPS configuration ('D-term' type) of the ABJM theory has been considered in [23]:

$$\frac{k}{2\pi} D_2 Z^I + Z^I (Z^\dagger Z - W W^\dagger) - (Z Z^\dagger - W^\dagger W) Z^I = 0, \tag{3.21}$$

$$-\frac{k}{2\pi} D_2 W^{I^\dagger} + W^{I^\dagger} (Z^\dagger Z - W W^\dagger) - (Z Z^\dagger - W^\dagger W) W^{I^\dagger} = 0, \tag{3.22}$$

and

$$F_I^\dagger := \frac{4\pi}{k} \epsilon_{IJ} \epsilon^{KL} W_K Z^J W_L = 0, \quad G^{\dagger I} := \frac{4\pi}{k} \epsilon^{IJ} \epsilon_{KL} Z^K W_J Z^L = 0. \tag{3.23}$$

If one identifies these equations with the boundary condition (3.19), one can deduce further details of the boundary potential $V_{\partial M}$. We obtain

$$V_{\partial M, D} = \frac{\pi}{k} \mathrm{Tr}[(Z Z^\dagger - W^\dagger W)^2 - (Z^\dagger Z - W W^\dagger)^2], \tag{3.24}$$

where the full potential can include an additional term which vanishes when (3.23) are imposed. Another BPS configuration ('F-term type') [23] is given by

$$\frac{k}{4\pi}D_2 Z^I - \epsilon^{IJ}\epsilon_{KL}W^{\dagger K}Z^{\dagger}_J W^{\dagger L} = 0, \tag{3.25}$$

$$\frac{k}{4\pi}D_2 W^{\dagger I} - \epsilon^{IJ}\epsilon_{KL}Z^K W_J Z^L = 0, \tag{3.26}$$

and

$$N^I := \sigma C^I - C^I\hat{\sigma} = 0, \quad I = 1, \cdots, 4, \tag{3.27}$$

where $\sigma := 2\pi/k(ZZ^\dagger - W^\dagger W)$, $\hat{\sigma} := 2\pi/k(Z^\dagger Z - WW^\dagger)$. This corresponds to the boundary potential

$$V_{\partial M,F} = -\frac{2\pi}{k}\mathrm{Tr}[\epsilon_{IJ}\epsilon^{KL}Z^I W_K Z^J W_L] + h.c., \tag{3.28}$$

where the full potential can include an additional term which vanishes when (3.27) is imposed. These boundary potentials have also been considered in [17]. Here we will argue that the full boundary potential is given by

$$V_{\partial M} := V_{\partial M,D} + V_{\partial M,F}. \tag{3.29}$$

To see this, it is sufficient that in principle one may add to (3.29) a term which vanishes when either (3.23) or (3.27) are imposed. However, combined with conformal invariance, there is no such quartic polynomial one can construct.

Before we move on, we remark that we have adopted above the definitions of [24] for $F_I, G^I, N^I, \sigma, \hat{\sigma}$. Using this notation, the potential for C^I in the ABJM theory can be written as

$$V_M = \mathrm{Tr}[F^\dagger_I F^I + G^{\dagger I}G_I] + \mathrm{Tr}[N^\dagger_I N^I]. \tag{3.30}$$

Next, let us consider the boundary actions (3.7) and (3.8) we have added to the bulk ABJM action. Obviously these cannot be the whole story as the fields g, $\hat{g}$ we added do not fully describe the bosonic content of a multiplet of $\mathcal{N} = (4,4)$ supersymmetry, which consists of 4 real scalar and 4 Weyl fermionic degrees of freedom. Therefore we need to supersymmetrize our boundary action by supplementing it with additional fields. We will show now how to do this.

The supersymmetric WZW action is a particular type of nonlinear sigma model in 2-dimensions. This type of nonlinear sigma model generalizes the original supersymmetric construction of nonlinear sigma models [25, 26] by utilizing in addition to a metric also a 2-form. The action is entirely determined by a flat connection whose torsion is determined by the 2-form [27–29]. The most general such manifolds are given by semisimple Lie groups. With respect to a basis of left invariant one-forms, the metric is simply given by the constant δ_{ab} and the torsion is given by the group

structure constants f_{abc}. The relative size between these two terms is fixed by supersymmetry. This gives rise to supersymmetric WZW models. These models always have $\mathcal{N}=(n,n)$ supersymmetry and $\mathcal{N}=(4,4)$ supersymmetry is the highest one can get. Moreover $\mathcal{N}=(3,3)$ implies $\mathcal{N}=(4,4)$. Therefore although we may only expect $\mathcal{N}=(3,3)$ for the full boundary ABJM theory, this WZW sector of the theory will actually have $\mathcal{N}=(4,4)$ supersymmetry. Futher conditions of supersymmetry impose additional contraints on the form of the group manifold. For $\mathcal{N}=(4,4)$, these are a particular type of quaternionic group manifolds. A list of all possible such group manifolds is given in table 1 of [27], or by taking products of factors there.

The $\mathcal{N}=(4,4)$ WZW theory we are after has $SU(2)\times SU(2)$ R-symmetry. It is instructive to recall how the R-symmetry is realized in the $\mathcal{N}=(4,4)$ models in general. Let us first consider the case of $\mathcal{N}=(3,3)$ WZW model which has a $SU(2)$ R-symmetry. The simplest such WZW model is given by [30]

$$S=\int \partial_+ u\partial_- u+\mathcal{L}_{WZW}(q_\alpha{}^\beta)+i\xi_+^{\alpha a}\partial_-\xi_{+\alpha a}+i\xi_-^{\alpha a}\partial_+\xi_{-\alpha a}, \tag{3.31}$$

where $u, q_\alpha{}^\beta$ are bosonic and $\xi_\pm$ are fermionic $U(1)$ fields and $\alpha,\beta=1,2$, $a,b=1,2$. The indices α,β are acted on by the R-symmetry $SU(2)_1$. Since we know $\mathcal{N}=(3,3)$ implies $\mathcal{N}=(4,4)$, there must be a second $SU(2)_2$ R-symmetry. Indeed, this is given by the $SU(2)_2$ which acts on the indicies a,b of the fermions. We note that this second $SU(2)_2$ cannot be seen if one considers only the bosonic sector. We also note that this pattern of R-symmetry enhancement is generic and applies to the more general situation where the fields are nonabelian. Now back to the $\mathcal{N}=(4,4)$ case. Since in general the group manifold G should contain the R-symmetry as a subgroup, in order to have an R-symmetry $SU(2)_1\times SU(2)_2$ which is fully visible in the bosonic sector, one has to consider a product group manifold $G=G_1\times G_2$ with $SU(2)_1$ acting on the factor G_1 and the $SU(2)_2$ acting on G_2. This product structure of the group manifold fits with our boundary degrees of freedom $g\in U(N)_1, \hat{g}\in U(N)_2$ introduced above. Inspecting table 1 of [27] suggests that six additional $U(N)$ degrees of freedom should be introduced in such a way that three of the new fields combine with g to form a group $G_1=U(2N)_1$ and the other three new fields combine with $\hat{g}$ to form a group $G_2=U(2N)_2$. That is, the required $\mathcal{N}=(4,4)$ supersymmetric WZW model should be based on the group manifold

$$G_1\times G_2=U(2N)\times U(2N). \tag{3.32}$$

This is the minimal group manifold with the property that the commutant of the R-symmetry is $U(N)\times U(N)$. Explicitly, one can denote the group elements $g\in G_1$, $\hat{g}\in G_2$ as

$$g=\exp(u\mathbf{1}+\varphi_i\sigma^i),\quad \hat{g}=\exp(\hat{u}\mathbf{1}+\hat{\varphi}_i\sigma^i), \tag{3.33}$$

where $\mathbf{1}$ is the identity 2×2 matrix and σ^i are the Pauli matrices. In this representation, the $SU(2)_1$ R-symmetry, for example, is represented by the generators

$$\mathbf{1}_{N\times N} \otimes \sigma_i \tag{3.34}$$

in $U(2N)_1$. The commutant subgroup is the one which one can gauge by coupling to the bulk ABJM gauge fields. Therefore it must contain $U(N) \times U(N)$. The choice (3.32) is a minimal choice because the commutant of the R-symmetry is exactly $U(N) \times U(N)$. We will comment on the possibility of other nonminimal choices of group manifold later.

Having successfully supersymmetrized the boundary WZW action to have the desired $\mathcal{N} = (4,4)$ supersymmetry, we now come to the issue of the cancellation of the gauge noninvariance of the boundary Chern-Simons action. It is easy to see that our previous construction can be carried out in exactly the same way by simply embedding $U(N)\times U(N)$ into $U(2N)\times U(2N)$, i.e. we tensor all the ABJM fields with $\mathbf{1}_{2\times 2} \times \mathbf{1}_{2\times 2}$. With this interpretation, the bosonic action of the $\mathcal{N} = (4,4)$ boundary ABJM theory with $U(N) \times U(N)$ gauge group is given by (3.10), with the definition of Tr including an additional normalization factor of 1/2. This obviously leaves the bulk theory unchanged. As for the boundary theory, the normalization of the WZW term is constrained by a topological argument [21]. Note that this constrains only the u and $\hat{u}$-part (as defined in (3.33)) and we do get the right normalization.

Some comments on this supersymmetrized action follow.

1. One can certainly consider other $\mathcal{N} = (4,4)$ group manifolds in which (3.32) is embedded, for example $G_1 \times G_2 = U(2M) \times U(2M)$ with large enough M [2]. In this case, the bulk plus boundary action can be obtained as a subsector of our construction with a bulk $U(M) \times U(M)$ ABJM theory.

2. In the above, we have supersymmetrized the WZW actions of the boundary actions (3.7), (3.8) with $\mathcal{N} = (4,4)$ supersymmetry. To render the whole boundary action supersymmetric, one still needs to supersymmetrize the gA and AA type of terms with the accompanying fermion action. We expect the resulting action will be superconformally invariant at the quantum level.

3. The coefficient of the kinetic term of g in the boundary action (3.7) was originally fixed by requiring a WZW action is formed so that one has conformal invariance. Here we see that the coefficient is also fixed by requiring supersymmetry of the nonlinear sigma model.

[2] All the groups in the table 1 of [27] can be embedded in $U(2M)$ for sufficiently large M except for the exceptional groups. However these can be included only as direct products which will decouple from the other fields.

3.3 Multiple self-dual strings action

To derive the action for the multiple self-dual strings, let us consider a bunch of open M2-branes suspended between two parallel M5-branes separated by a distance $x^2 = L$. Denoting the string worldsheet coordinates by (σ_0, σ_1), one has to take a limit such that all the physical configurations on the M2-branes are independent of σ_2. This corresponds to suppression of the membranes modes of motion. This can be achieved by taking all the gauge fields to be independent of σ_2 and the bulk matter fields to be independent of σ_2 covariantly:

$$\partial_2 A_M = 0, \quad D_2 C^I = 0, \tag{3.35}$$

The fact that the M2-branes are ending on an M5-brane means that the four fields C^I should be divided into two groups, with two C's describing the longitudinal directions on the M5-brane, and two C's describing the directions transverse to the M5-brane. Denoting the latter as $W^I \propto \mathbf{1}$ and the former as Z^I, $I = 1, 2$, the C-dependent part of the action (3.10) becomes

$$-L \int d^2x \left[|DZ|^2 + V_M(Z) + \frac{1}{L} V_{\partial M}(Z) \right]. \tag{3.36}$$

Here we have taken a gauge $A_2^{(i)} = 0$ so that C^I are independent of σ_2 and (3.36) makes sense.

For the Chern-Simons terms, it is easy to see that there is nothing left after dropping derivatives with respect to σ_2 and setting $A_2^{(i)} = 0$. As for the WZW terms, since the M2-branes end on two different M5-brane, we get an $\mathcal{N} = (4, 4)$ supersymmetric $U(2N) \times U(2N)$ WZW action on each boundary. Consider one of the boundaries, say ∂M_1. Adding the boundary action to (3.36), we obtain the action for N multiple self-dual strings living on the boundary ∂M_1,

$$\begin{aligned} &-L \int \left[|DZ|^2 + V_M(Z) + \frac{1}{L} V_{\partial M}(Z) \right] \\ &+S^{(-)}_{WZW}[g] + \frac{k}{4\pi} \int \partial_+ g g^{-1} A^{(1)}_- - \frac{k}{8\pi} \int A^{(1)\,2}_\mu \\ &+S^{(+)}_{WZW}[\hat{g}] - \frac{k}{4\pi} \int \partial_- \hat{g} \hat{g}^{-1} A^{(2)}_+ - \frac{k}{8\pi} \int A^{(2)\,2}_\mu + \text{fermions}. \end{aligned} \tag{3.37}$$

We note that since $A^{(1)}_\pm$ and $A^{(2)}_\pm$ appear linearly in the action, one may try to integrate out, for example, $A^{(1)}_-$ and $A^{(2)}_+$ and obtain constraints between $A^{(1)}_+$, $A^{(2)}_-$ with Z, g and $\hat{g}$. Solving these constraints for $A^{(1)}_+$ and $A^{(2)}_-$ and substituting back into (3.37) would give us an action given in terms of Z, g and $\hat{g}$ only. The constraints are however complicated and solving it involves nonlocal expressions. So it is better

to present the action in the form (3.37). We also note that the introduction of the scale L breaks the conformal symmetry of (3.10) and the action (3.37) describes self-dual strings configurations that are at energy scale below $1/L$. It is interesting to ask what (3.37) flows to at lower energies. Addressing this is beyond the scope of this paper.

4. Discussion

In this paper we have provided a new method to treat a twisted Chern-Simons matter system with boundary. The guiding principle in our construction is the manifest preservation of gauge symmetry and conformal symmetry of the system including the boundary. By applying our method to the ABJM theory with boundary, and together with the requirement of respecting the $\mathcal{N} = (4, 4)$ supersymmetry multiplet structure, we identified the new degrees of freedom g and $\hat{g}$ that must be present on the worldsheet of multiple self-dual strings. These degrees of freedom generate a $U(2N) \times U(2N)$ Kac-Moody current algebra on the worldsheet. It will be interesting to understanding better the role of these currents in the physics of multiple self-dual strings. For example, these currents could tell us something about intersecting brane configurations on M5-branes. It is also an open question whether these currents couple to non-abelian gauge bosons in the background.

Another result we obtain is the determination of the boundary potential $V_{\partial M}$ with the use of supersymmetry and the scaling property of the potential. It would be interesting to consider the BPS equations for (3.37) and study the properties of other solitons within M5-branes.

In this paper, we have considered a system of free self-dual strings. It will be interesting to include couplings to background fields on the M5-brane. One particularly interesting background is the non-abelian self dual 2-form potential which is expected to arise when there are multiple M5-branes. The description of a non-abelian tensor is an open problem. An interesting attempt has been considered in [31] which involves the use of a loop space. The origin of this "extra dimension" is however not clear. One can speculate that this is related to the difficulty of quantizing the membrane which has a continuous spectrum. Consequently this may lead to new issues in reducing the membrane action to the string action.

In [17], chiral WZW actions with opposite signs of kinetic terms were obtained on the boundary of the open M2-branes theory. It was argued that a parity operation can be defined which exchanges the two kinetic terms of the WZW actions, resulting in a nonchiral theory. However this parity operation does not address the issue of the ill-defined kinetic terms. In our construction, we obtain nonchiral WZW actions immediately. Moreover our WZW action has well defined kinetic terms.

By dimensional reduction, one can obtain the action for multiple self-dual strings on the NS5-brane. This action would seem to be different from what one would obtain by considering D2-branes suspended between NS5-branes. In particular, the D2-branes non-abelian Born-Infeld action does not contain any Chern-Simons terms and so it would not appear necessary to introduce the boundary degrees of freedom g and $\hat{g}$. However we predict that the WZW action will still arise in the boundary theory. It would be interesting to understand this issue properly, as well as the reduction to the D4-brane system.

Finally we comment that one may also consider having a set of N M2-branes ending on two separated M9-branes as in the Horava-Witten setup [32]. We expect the $E_8 \times E_8$ gauge symmetry will arise from the gravitational anomaly in the same way as in [32]. The Kac-Moody symmetry on the worldsheet is however new and seems to suggest the emergence of additional gauge symmetry in spacetime. This is entirely unclear. It is important to understand better the role of the Kac-Moody symmetry, both for the multiple self-dual strings on M5-brane and for the multiple heterotic strings, which is one of the main predictions of our construction.

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